Problem Set 3 – Solutions

Problem 1

	$f(x)$	$f'(x)$
1.	$10x$	10
2.	$4x-5$	4
3.	$100x-3x^2$	$100-6x$
4.	$-10x+6x^2-\frac{2}{3}x^3$	$-10+12x-2x^2$
5.	$5x^{1/3}$	$\frac{5}{3}x^{1/3-1}=\frac{5}{3}x^{-2/3}$
6.	$\frac{1}{x^{1/5}}=x^{-1/5}$	$-\frac{1}{5}x^{-1/5-1}=-\frac{1}{5}x^{-6/5}$
7.	$e^{-\frac{1}{2}x}$	$-\frac{1}{2}e^{-\frac{1}{2}x}$
8.	$\ln(1/x)=-\ln(x)$	$-\frac{1}{x}$

Problem 2

1. $$\int_0^1 e^{-0.1t}\,dt = -\frac{1}{0.1}e^{-0.1t}\Big|_0^1$$
$$= 10\,(1-e^{-0.1}) \simeq 0.9516$$

2. $$\int_0^1 3x^5\,dx = 3\,\frac{x^6}{6}\Big|_0^1 = \frac{3}{6} = \frac{1}{2}$$

3. $$\int_0^1 \frac{1}{x+5}\,dx = \ln(x+5)\Big|_0^1 = \ln 6 - \ln 5 \simeq 0.1823$$

Problem 3

a. $$V_0 = \int_0^T c e^{gs} e^{-rs} ds$$
$$= \int_0^T c e^{-(r-g)s} ds$$
$$= c \left[-\frac{1}{r-g} e^{-(r-g)s} \Big|_0^T \right]$$
$$= \frac{c}{r-g} \left(1 - e^{-(r-g)T}\right)$$

b. $$V_t = \int_t^T c e^{gs} e^{-r(s-t)} ds$$
$$= c e^{rt} \int_t^T e^{-(r-g)s} ds$$
$$= c e^{rt} \left[-\frac{1}{r-g} e^{-(r-g)s} \Big|_t^T \right]$$
$$= c e^{rt} \frac{1}{r-g} \left(e^{-(r-g)t} - e^{-(r-g)T} \right)$$

c. $$V_T = \int_T^T c e^{gs} e^{-r(s-T)} ds = 0.$$

<u>Notes</u>: In b., the cash flow at time $s > t$ is $c e^{gs}$ but the discounting is from s to t, hence $e^{-r(s-t)}$.

Problem 4

$$dS = \mu S\,dt + \sigma S\,dz$$

$$Y = S^4 \qquad Y_S = 4S^3 \qquad Y_{SS} = 12S^2$$

$$dY = Y_S\,dS + \frac{1}{2}Y_{SS}(dS)^2$$

$$= Y_S(\mu S\,dt + \sigma S\,dz) + \frac{1}{2}Y_{SS}\,\sigma^2 S^2\,dt$$

$$= 4S^3(\mu S\,dt + \sigma S\,dz) + \frac{1}{2}12S^2\sigma^2 S^2\,dt$$

$$= 4\mu S^4\,dt + 4\sigma S^4\,dz + 6\sigma^2 S^4\,dt$$

$$= (4\mu + 6\sigma^2)S^4\,dt + 4\sigma S^4\,dz$$

$$= (4\mu + 6\sigma^2)Y\,dt + 4\sigma Y\,dz$$

or $$\frac{dY}{Y} = (4\mu + 6\sigma^2)\,dt + 4\sigma\,dz.$$

Problem 5

$$dS = \mu S\,dt + \sigma S\,dz$$

$Y = S e^{-\mu t}$ $\quad Y_S = e^{-\mu t}$ $\quad Y_{SS} = 0$ $\quad Y_t = -\mu S e^{-\mu t}$

$$\begin{aligned}
dY &= Y_S\,dS + \tfrac{1}{2} Y_{SS} (dS)^2 + Y_t\,dt \\
&= e^{-\mu t} (\mu S\,dt + \sigma S\,dz) - \mu S e^{-\mu t}\,dt \\
&= \mu S e^{-\mu t}\,dt + \sigma S e^{-\mu t}\,dz - \mu S e^{-\mu t}\,dt \\
&= \sigma S e^{-\mu t}\,dz \\
&= \sigma Y\,dz
\end{aligned}$$

So Y follows a geometric Brownian motion with drift zero and volatility σ.

Problem 6

$$dS = rS\,dt + \sigma S\,dz$$

$$F = S e^{r(T-t)}$$

$$F_S = e^{r(T-t)} \qquad F_{SS} = 0 \qquad F_t = -rSe^{r(T-t)}$$

$$\begin{aligned} dF &= F_S\,dS + \frac{1}{2}F_{SS}(dS)^2 + F_t\,dt \\ &= e^{r(T-t)}(rS\,dt + \sigma S\,dz) - rSe^{r(T-t)}dt \\ &= rSe^{r(T-t)}dt + \sigma S e^{r(T-t)}dz - rSe^{r(T-t)}dt \\ &= \sigma S e^{r(T-t)}dz \\ &= \sigma F\,dz \end{aligned}$$

We proved in class that a GBM with constant volatility σ and no drift is a martingale.

Note: This was not required to solve the problem, but we have that

$$F_0 = E\,F_t \quad \text{so that} \quad F_0 = E\,F_T = E(S_T)$$

The current futures price is given by the expected stock price under the risk-neutral measure.